ON THE SIMPLE GROUPS WHICH CAN BE REPRESENTED AS SUBSTITUTION GROUPS THAT CONTAIN CYCLICAL SUBSTITUTIONS OF A PRIME DEGREE.

By DR. G. A. MILLER.

It is known that cyclical substitutions of a prime degree (p) cannot occur in any primitive group (if it is not alternating or symmetric) unless the degree of the primitive group (G) is one of the following three numbers, p, $p+1$, $p+2$. When G is of degree p all its substitutions of order p generate a simple group which is a selfconjugate subgroup of G (if it does not coincide with G) and corresponds to identity of a cyclical quotient group of G, the order of this quotient group being a divisor of $p-1$.* We proceed to consider the cases when G contains substitutions of order p and is either of degree $p+1$ or of degree $p+2$.

If G is of degree $p+2$ and is not generated by its substitutions of order p it must contain a selfconjugate subgroup (G') which is generated by these substitutions, G' is at least triply transitive since it contains substitutions of order p.† We proceed to prove that it is a simple group when p exceeds 2. If it were compound each of its at least doubly transitive maximal subgroups of degree $p+1$ would also be compound, since the substitutions of such a subgroup (G_1) which would belong to the given selfconjugate subgroup of G, would form a selfconjugate subgroup of G_1.

A maximal subgroup (G_2) of degree p that is contained in G_1 must be transitive. Since every selfconjugate subgroup of a transitive group of degree p must contain substitutions of order p, G_2 could have only identity in common with the given self-conjugate subgroup of G. Hence the order of this selfconjugate subgroup would be $(p+1)(p+2)$, and the corresponding quotient group would be simply isomorphic to G_2. To a subgroup of order p that is contained in G_2 there would correspond a subgroup of G, whose order would be $p(p+1)(p+2)$. Since this is evidently impossible we have the important

THEOREM. *All the substitutions of order p (p being any odd prime number) that are contained in any primitive group of degree p or of degree $p+2$ generate a simple group. If this simple group does not coincide with the entire group it is selfconjugate and the corresponding quotient group is cyclical and has for its order a divisor of $p-1$.‡ This simple group cannot be selfconjugate subgroup of more than one group of its own degree and of a given order.*

We shall now consider the groups of degree $p+1$, p being any odd prime number, that contain substitutions of order p. Such groups are at least doubly transitive and their substitutions of order p generate a doubly transitive selfconjugate subgroup if they do not generate the entire group. Let H be such a sub-

*Cf. *Bulletin of the American Mathematical Society*, Vol. 4, 1898, page 140.

†Jordan, *Journal de Mathematiques*, Vol. 16, 1871.

‡From this theorem it follows directly that the primitive group of degree 9 and of order 504 is simple.

group. We see as in the preceding paragraph that H cannot contain any selfconjugate subgroup unless this subgroup is a regular group of order $p+1$. As this regular subgroup has to contain p subgroups that are conjugate in H it cannot involve any substitution whose order exceeds 2. Hence we have the

THEOREM. *If $p+1$ is not a power of 2 then the substitutions of order p (p being any odd prime number) that are contained in a group of degree $p+1$ generate a simple group. If this simple group does not coincide with the entire group it is selfconjugate and the corresponding quotient group is a cyclical and its order is a divisor of $p-1$.* If $p+1$ is a power of 2, the group generated by the substitutions of order p that are contained in a group of degree $p+1$ cannot contain any selfconjugate subgroup except perhaps the regular group of order $p+1$ which contains no substitution whose order exceeds 2.*

Cornell University, April 3, 1899.

*From this theorem it follows directly that the three primitive groups of degree 12 and orders 660, 7920, and 95040, respectively, are simple. The last of these three groups is the well known five-fold transitive group of Mathien.

ON SYMMETRIC FUNCTIONS.

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[Continued from March Number.]

B. FUNDAMENTAL RELATIONS FOR SYMMETRIC FUNCTIONS.

1. FUNDAMENTAL RELATIONS BETWEEN COEFFICIENTS.

(1). *Derivation of the relations.*

In A, 4, (3) we have already obtained one of the relations, viz:

$$\begin{pmatrix} 0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots\ldots n^{\lambda_n} \\ b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_m{}^{\kappa_n}\end{pmatrix}=(-1)^{mn}\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n) \\ a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0\end{pmatrix}.$$

If in the equations $b_0x^n+b_1x^{n-1}+\ldots\ldots+b_n'=0$, and $a_0x^m+a_1x^{m-1}+\ldots\ldots+a_m=0$ (cf. loc. cit.), we substitute $x=1/y$, b_r becomes b_{n-r}, a_r becomes a_{m-r}, and $b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n}$ becomes

$$b_n{}^m\Sigma\frac{1}{\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n}}=\frac{b_n{}^m}{(\beta_1\beta_2\ldots\ldots\beta_n)^m}\Sigma\beta_1{}^{m-\kappa_1}\beta_2{}^{m-\kappa_2}\ldots\ldots\beta_n{}^{m-\kappa_n}$$

$$=(-1)^{mn}b_0{}^m\Sigma\beta_1{}^{m-\kappa_1}\beta_2{}^{m-\kappa_2}\ldots\ldots\beta_n{}^{m-\kappa_n}.$$

Similarly $a_0{}^n\Sigma(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots(\alpha\lambda_n)^0$ becomes $(-1)^{mn}a_0{}^n\Sigma(\alpha\lambda_n)^n(\alpha\lambda_{n-1})^{n-1}\ldots\ldots(\alpha\lambda_0)^0$. We have therefore,